

Geometrically Underpinned Maximally Entangled States Bases

M. Revzen¹

¹*Department of Physics, Technion - Israel Institute of Technology, Haifa 32000, Israel*

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Finite geometry is used to underpin finite, two d-dimensional particles Hilbert space, $d=\text{prime} \neq 2$. A central role is allotted to states with mutual unbiased bases (MUB) labeling. Dual affine plane geometry (DAPG) *points* underpin single particle, MUB labeled, product states. The DAPG *lines* are shown to underpin maximally entangled states which form an orthonormal basis spanning the space. The relevance of mutually unbiased collective coordinates bases (MUCB) for dealing with maximally entangled states is discussed and shown to provide an economic alternative mode of study. These maximally entangled, geometrically reasoned states, provide the resource to a transparent solution to what may be termed tracking of the Mean King Problem (MKP): here Alice prepares a state measured by King along some orientation which Alice succeed in identifying with a subsequent measurement. Brief expositions of the topics considered: MUB, DAPG, MUCB and the MKP are included, rendering the paper self contained.

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I. INTRODUCTION

Several recent studies [1, 3–7, 12, 15, 16] consider the affinity of finite, d , dimensional Hilbert space to finite Galois fields, $\text{GF}(d)$, and thereby to finite geometry. These interrelations are of interest as they illuminate both subjects. The present work contains a novel intuitive geometrical underpinning for the MUB structure of d^2 dimensional Hilbert space accommodating two d-dimensional particles. ($d=\text{prime} (\neq 2)$.) The study gives for the first time, to our knowledge, explicit formulae that relates lines and points of the geometry to *states* (rather than projectors) [17] allowing a geometric view of the relation between product and maximally entangled states [23].

The analysis underpins Hilbert space states (and operators) with geometrical points and lines. In particular we introduce, in section IV, a simple, i.e. universal, balancing term. This term, denoted by \mathcal{R} , arises upon the association of states addition in Hilbert space with geometrical requirements among points and lines. (The corresponding term is the unit operator in studies, [1, 18], wherein the association was to Hilbert space projectors [1, 17].) Its origin, given in the uncanny affinity of mutual unbiased bases (MUB) labeling for a geometrical coordination scheme, is outlined in section IV. Included in this section is a brief exposition of DAPG which we use to underpin a d-dimensional single particle Hilbert space state projectors [1, 2, 12, 17, 28] the results of which are used in the present analysis that pertains to a d^2 dimensional Hilbert spaces.

In section V we give the central result of this paper i.e. the demonstration that the state underpinned with geometrical line is a maximally entangled state of remarkable attribute: its overlap with (judicially defined to relate to one coordinate point) two particles product state is definitive. Thus it is $1/d$ if the underpinning point is on the line, nil otherwise. Since d lines share a point this relates to d lines. This holds while the constituent single particle state projectors have nonvanishing overlap with each of the d^2 (orthogonal maximally) entangled states that span the space. This issue is elaborated on in section VIII and leads to a novel tracking of the Mean King problem outlined in section IX : Alice produces a state which allow the tracking of the alignment of the King apparatus used in his measurement of the state by one subsequent measurement.

An alternative approach to the construction of a d^2 dimensional maximally entangled states basis is given in section VII where these states are shown to be *product* states of *collective* coordinates MUB bases. The theory of this approach is outlined in sections II and III. The account via the collective MUB states proves to be more economic and, perhaps, more informative physically.

II. FINITE DIMENSIONAL MUTUAL UNBIASED BASES, (MUB): BRIEF REVIEW

In a d-dimensional Hilbert space, two complete, orthonormal vectorial bases, \mathcal{B}_1 , \mathcal{B}_2 , are said to be MUB if and only if ($\mathcal{B}_1 \neq \mathcal{B}_2$)

$$\forall |u\rangle, |v\rangle \in \mathcal{B}_1, \mathcal{B}_2 \text{ resp., } |\langle u|v\rangle| = 1/\sqrt{d}. \quad (2.1)$$

The physical meaning of this is that knowledge that a system is in a particular state in one basis implies complete ignorance of its state in the other basis.

Ivanovich [25] proved that there are at most $d+1$ MUB, pairwise, in a d -dimensional Hilbert space and gave an explicit formulae for the $d+1$ bases in the case of $d=p$ (prime number). Wootters and Fields [8] constructed such $d+1$ bases for $d = p^m$ with m an integer. Variety of methods for construction of the $d+1$ bases for $d = p^m$ are now available [2, 9, 26, 28]. Our present study is confined to $d = p \neq 2$.

We now give explicitly the MUB states in conjunction with the algebraically complete operators [14, 24] set: \hat{Z}, \hat{X} . Thus we label the d orthonormal states spanning the Hilbert space, termed the computational basis, by $|n\rangle$, $n = 0, 1, \dots, d-1$; $|n+d\rangle = |n\rangle$

$$\hat{Z}|n\rangle = \omega^n|n\rangle; \hat{X}|n\rangle = |n+1\rangle, \omega = e^{i2\pi/d}. \quad (2.2)$$

The d states in each of the $d+1$ MUB bases [14, 26] are the states of the computational basis and the d bases:

$$|m; b\rangle = \frac{1}{\sqrt{d}} \sum_0^{d-1} \omega^{\frac{b}{2}n(n-1)-nm}|n\rangle; \quad b, m = 0, 1, \dots, d-1. \quad (2.3)$$

Here the d sets labeled by b are the bases and the m labels the states within a basis. We have [26]

$$\hat{X}\hat{Z}^b|m; b\rangle = \omega^m|m; b\rangle. \quad (2.4)$$

For later reference we shall refer to the computational basis (CB) by $b = \ddot{0}$. Thus the above gives $d+1$ bases, $b = \ddot{0}, 0, 1, \dots, d-1$ with the total number of states $d(d+1)$ grouped in $d+1$ sets each of d states. We have of course,

$$\langle m; b|m'; b\rangle = \delta_{m,m'}; \quad |\langle m; b|m'; b'\rangle| = \frac{1}{\sqrt{d}}, \quad b \neq b'. \quad (2.5)$$

The MUB set is closed under complex conjugation:

$$\langle n|m, b\rangle^* = \langle n|\tilde{m}, \tilde{b}\rangle, \Rightarrow |\tilde{m}, \tilde{b}\rangle = |d-m, d-b\rangle, \quad (2.6)$$

as can be verified by inspection of Eq.(2.3). (We denote $(m, \ddot{0})$ by \tilde{m} when no confusion should arise.)

This completes our discussion of single particle MUB.

III. MUB FOR COLLECTIVE COORDINATES

Several studies [12, 19, 20, 23, 27] consider the entanglement of two d -dimensional particles Hilbert space via MUB state labeling. We shall now outline briefly the approach adopted by [23] that will be used in later sections. The Hilbert space is spanned by the single particle computational bases, $|n\rangle_1|n'\rangle_2$ (the subscripts denote the particles). These are eigenfunctions of \hat{Z}_i $i=1,2$: $\hat{Z}_i|n\rangle_i = \omega^n|n\rangle_i$, $\omega = e^{i\frac{2\pi}{d}}$. Similarly $\hat{X}_i|n\rangle_i = |n+1\rangle$, $i = 1, 2$. We now define our collective coordinates and collective operators (we remind the reader that the exponents are modular variables, e.g. $1/2 \bmod[d=7]=(d+1)/2=4$):

$$\bar{Z}_r \equiv \hat{Z}_1^{1/2} \hat{Z}_2^{-1/2}; \quad \bar{Z}_c \equiv \hat{Z}_1^{1/2} \hat{Z}_2^{1/2}. \quad (3.1)$$

Since $\bar{Z}_s^d = 1$, $s = r, c$ we may consider their respective computational eigen-bases,

$$\bar{Z}_s|n\rangle_s = \omega^n|n\rangle_s, \quad n = 0, 1, \dots, d-1, \quad s = r, c. \quad (3.2)$$

clearly $|n\rangle_r|n'\rangle_c$; $n, n' = 0, 1, \dots, d-1$, is a d^2 orthonormal basis spanning the two d -dimensional particles Hilbert space. Eq.(3.1) implies,

$$\hat{Z}_1 = \bar{Z}_r \bar{Z}_c; \quad \hat{Z}_2 = \bar{Z}_r^{-1} \bar{Z}_c \quad (3.3)$$

In a similar fashion we define the displacement operators,

$$\bar{X}_r \equiv \hat{X}_1 \hat{X}_2^{-1}; \quad \bar{X}_c \equiv \hat{X}_1 \hat{X}_2 \rightarrow \hat{X}_1 = \bar{X}_r^{1/2} \bar{X}_c^{1/2}, \quad \hat{X}_2 = \bar{X}_r^{-1/2} \bar{X}_c^{1/2}. \quad (3.4)$$

These entail

$$\bar{X}_s \bar{Z}_s = \omega \bar{Z}_s \bar{X}_s, \quad s = r, c; \quad \bar{X}_s \bar{Z}_{s'} = \bar{Z}_{s'} \bar{X}_s, \quad s \neq s'. \quad (3.5)$$

One readily proves [23],

$$\langle n_1, n_2 | n_r, n_c \rangle = \delta_{n_r, (n_1 - n_2)/2} \delta_{n_c, (n_1 + n_2)/2}. \quad (3.6)$$

Note that the formula is schematic. Thus $|n_1, n_2\rangle = |n\rangle_1 |n'\rangle_2$, $|n_r, n_c\rangle = |n\rangle_r |n'\rangle_c$ i.e. they refer to distinct bases. We have then,

$$|n_r, n_c\rangle = |n_1, n_2\rangle, \text{ for } n_r = (n_1 - n_2)/2, n_c = (n_1 + n_2)/2 \Leftrightarrow n_1 = n_r + n_c, n_2 = n_c - n_r. \quad (3.7)$$

Thence we may consider collective MUB,

$$|m_s; b_s\rangle = \frac{1}{\sqrt{d}} \sum_n \omega^{\frac{b_s}{2} n(n-1) - m_s n} |n\rangle_s, \quad m_s, b_s = 0, 1, \dots, d-1; \quad s = r, c. \quad (3.8)$$

Incorporating the respective CB $b_s = \ddot{0}_s$ it is proved in [23] that the two d-dimensional particles state,

$$|m_r; b_r\rangle |m_c; b_c\rangle \text{ for } b_r \neq b_c \quad (3.9)$$

is a maximally entangled state. (For $b_r = b_c$ it is a product state for both *particles* and collective coordinates.) Indeed that it is a maximally entangled state may be seen by tracing out the first particle coordinates,

$$\sum_{n=0}^{d-1} \langle n | \ddot{m} \rangle_c \langle 2m_0 |_r \langle \ddot{m} |_c | n \rangle_1 = \sum_{n, n_r, n_r'} \langle n | \ddot{m} + n_r \rangle | \ddot{m} - n_r \rangle_2 \omega^{-2n_r m_0} \langle \ddot{m} - n_r' |_2 \langle \ddot{m} + n_r' | n \rangle \omega^{2n_r' m_0} = \frac{1}{d} \hat{I}_2. \quad (3.10)$$

where we used Eqs.(3.7,3.8).

This completes our review of mutual collective unbiased bases (MUCB).

IV. FINITE GEOMETRY AND HILBERT SPACE OPERATORS

We now briefly review the essential features of finite geometry required for our study [1, 10, 11, 29].

A finite plane geometry is a system possessing a finite number of points and lines. There are two kinds of finite plane geometry: affine and projective. We shall confine ourselves to affine plane geometry (APG) which is defined as follows. An APG is a non empty set whose elements are called points. These are grouped in subsets called lines subject to:

1. Given any two distinct points there is exactly one line containing both.
2. Given a line L and a point S not in L ($S \notin L$), there exists exactly one line L' containing S such that $L \cap L' = \emptyset$. This is the parallel postulate.

3. There are 3 points that are not collinear.

It can be shown [10, 29] that for $d = p^m$ (a power of prime) APG can be constructed (our study here is for $d=p$). Furthermore The existence of APG implies [10, 29] the existence of its dual geometry DAPG wherein the points and lines are interchanged. Since we shall study extensively DAPG, we list its properties [10, 29]. We shall refer to these by DAPG(\cdot):

- a. The number of lines is d^2 , L_j , $j = 1, 2, \dots, d^2$. The number of points is $d(d+1)$, S_α , $\alpha = 1, 2, \dots, d(d+1)$.
- b. A pair of points on a line determine a line uniquely. Two (distinct) lines share one and only one point.
- c. Each point is common to d lines. Each line contain $d+1$ points.
- d. The $d(d+1)$ points may be grouped in sets of d points, no two of a set share a line. Such a set is designated by $\alpha' \in \{\alpha \cup M_\alpha\}$, $\alpha' = 1, 2, \dots, d$. (M_α contain all the points not connected to α - they are not connected among themselves.) i.e. such a set contains d disjoint (among themselves) points. These are equivalent classes of the geometry [10]. There are $d+1$ such sets:

$$\bigcup_{\alpha=1}^{d(d+1)} S_\alpha = \bigcup_{\alpha=1}^d R_\alpha; \quad R_\alpha = \bigcup_{\alpha' \in \alpha \cup M_\alpha} S_{\alpha'}; \quad R_\alpha \cap R_{\alpha'} = \emptyset, \quad \alpha \neq \alpha'.$$

- e. Each point of a set of disjoint points is connected to every other point not in its set.

DAPG(c) allows the definition, which we adopt, of S_α in terms of addition of L_j which acquires a meaning upon

viewing the points (S_α) and the lines (L_j) as underpinning Hilbert space entities (e.g. projectors or states, to be specified later):

$$S_\alpha = \frac{1}{d} \sum_{j \in \alpha} L_j. \quad (4.1)$$

This implies,

$$\sum_{\alpha' \in \alpha \cup M_\alpha} S_{\alpha'} = \frac{1}{d} \sum_j L_j, \quad (4.2)$$

DAPG(d) via Eq.(4.2) implies,

$$\mathcal{R} = \sum_{\alpha' \in \alpha \cup M_\alpha} S_{\alpha'} = \frac{1}{d} \sum_j L_j = \frac{1}{d+1} \sum_\alpha S_\alpha \text{ independent of } \alpha. \quad (4.3)$$

This equation, Eq(4.3), reflects relation among equivalent classes within the geometry [10]. It will be referred to as the balance formula: the quantity \mathcal{R} serves as a balancing term. Thus, Eqs.(4.1),(4.3) imply,

$$L_j = \sum_{\alpha \in j} S_\alpha - \mathcal{R}. \quad (4.4)$$

(Note that in previous studies, [1, 17], where the geometrical point S_α underpins the projector, $S_\alpha \rightarrow \hat{A}_{\alpha=m,b} \equiv |m,b\rangle\langle b,m|$ gives $\mathcal{R} = \mathbb{I}$, i.e. independent of α .) A particular arrangement of lines and points that satisfies DAPG(x), x=a,b,c,d,e is referred to as a realization of DAPG. We outline in Appendix A the reasoning and proofs for the geometrically based interrelation among the geometrically underpinned Hilbert space operators. This completes our review of finite geometry.

V. REALIZATION OF DAPG

We now consider a particular realization of DAPG of dimensionality $d = p \neq 2$ which is the basis of our present study. We arrange the aggregate, the $d(d+1)$ points, α , in a $d \cdot (d+1)$ matrix like rectangular array of d rows and $d+1$ columns. Each column is made of a set of d points $R_\alpha = \bigcup_{\alpha' \in \alpha \cup M_\alpha} S_{\alpha'}$; (DAPG(d)). We label the columns by $b=\bar{0}, 0, 1, 2, \dots, d-1$ and the rows by $m=0, 1, 2, \dots, d-1$. (Note that the first column label of $\bar{0}$ is for convenience and does not relate to a numerical value. It designates the computational basis, CB.) Thus $\alpha = m(b)$ denotes a point by its row, m , and its column, b ; when b is allowed to vary - it gives the point's row position in every column defining thereby the line.. We label the left most column by $b=\bar{0}$ and with increasing values of b , that relates to the basis label, we move to the right. Thus the right most column is $b=d-1$. The top most point in each column is labeled by $m=0$ with m values increasing as one move to lower rows - the bottom row being $m=d-1$.

e.g. for $d=3$ the underpinning's schematics is:

$$\begin{pmatrix} m \backslash b & \bar{0} & 0 & 1 & 2 \\ 0 & A_{(0,\bar{0})} & A_{(0,0)} & A_{(0,1)} & A_{(0,2)} \\ 1 & A_{(1,\bar{0})} & A_{(1,0)} & A_{(1,1)} & A_{(1,2)} \\ 2 & A_{(2,\bar{0})} & A_{(2,0)} & A_{(2,1)} & A_{(2,2)} \end{pmatrix}$$

(In the Hilbert space realization of DAPG, A stands for the Hilbert space entity being underpinned with coordinated point, (m,b) . In [17] A represented an MUB projector: $A_{\alpha=(m,b)} = \hat{A}_\alpha = |m,b\rangle\langle b,m|$. In the present paper A will be seen to signify a two particles state to be specified in a subsequent section.) We now assert that the $d+1$ points, $m_j(b)$, $b = 0, 1, 2, \dots, d-1$, and $m_j(\bar{0})$, that form the line j which contain the two (specific) points $m(\bar{0})$ and $m(0)$ is given by (we forfeit the subscript j - it is implicit),

$$m(b) = \frac{b}{2}(2m(\bar{0}) - 1) + m(0), \text{ mod}[d] \quad b \neq \bar{0}; \quad m(\bar{0}) = \bar{m}. \quad (5.1)$$

The rationale for this particular form is clarified in the next section. Thus a line j is parameterized fully by $j = (m(\ddot{0}), m(0))$. We now prove that the set $j = 1, 2, 3 \dots d^2$ lines covered by Eq.(5.1) with the points as defined above forms a realization of DAPG.

1. Since each of the parameters, $m(\ddot{0})$ and $m(0)$, can have d values - the number of lines d^2 . The number of points in a line is evidently $d+1$ - one in each column: The linearity of the equation precludes having two points with a common value of b on the same line. DAPG(a).

2. Consider two points on a given line, $m(b_1), m(b_2)$; $b_1 \neq b_2$. We have from Eq.(5.1), ($b \neq \ddot{0}$, $b_1 \neq b_2$)

$$m(b_i) = \frac{b_i}{2}(2m(\ddot{0}) - 1) + m(0), \quad \text{mod}[d] \quad i = 1, 2. \quad (5.2)$$

These two equation determine uniquely (for $d=p$, prime) $m(\ddot{0})$ and $m(0)$. DAPG(b).

For fixed point, $m(b)$, \ddot{m} and $m(0)$ are interrelated, $\ddot{m} \Leftrightarrow m(0)$, thus the number of free parameters is d (the number of points on a fixed column). Thus each point is common to d lines. That the line contain $d+1$ is obvious. DAPG(c).

3. As is argued in 1 above no line contain two points in the same column (i.e. with equal b). Thus the d points, α , in a column form a set $R_\alpha = \bigcup_{\alpha' \in \alpha \cup M_\alpha} S_{\alpha'}$, with trivially $R_\alpha \cap R_{\alpha'} = \emptyset$, $\alpha \neq \alpha'$, and $\bigcup_{\alpha=1}^{d(d+1)} S_\alpha = \bigcup_{\alpha=1}^d R_\alpha$. DAPG(d).

4. Consider two arbitrary points *not* in the same set, R_α defined above: $m(b_1), m(b_2)$ ($b_1 \neq b_2$). The argument of 2 above states that, for $d=p$, there is a unique solution for the two parameters that specify the line containing these points. DAPG(e).

For example the point $m(1)$ is gotten from, Eq.(5.1),

$$m(1) = \frac{1}{2}(2 - 1) + 2 = 1 \quad \text{mod}[3] \Rightarrow m(1) = (1, 1).$$

We illustrate the above for $d=3$. The line j labeled by $j = (\ddot{m}, m(0))$ is made up of the 4 points j : 1. $(m(\ddot{0}) = (1, \ddot{0})$; 2. $m(0) = (2, 0)$; 3. $m(1) = (1, 1)$ and 4. $m(2) = (0, 2)$. (We shall denote $m(\ddot{0})$ by \ddot{m} when no confusion should arise.) The bracketed numbers give the point's coordinates.

$$\begin{pmatrix} m \backslash b & \ddot{0} & 0 & 1 & 2 \\ 0 & \cdot & \cdot & \cdot & (0, 2) \\ 1 & (1, \ddot{0}) & \cdot & (1, 1) & \cdot \\ 2 & \cdot & (2, 0) & \cdot & \cdot \end{pmatrix}$$

In [17, 18] we considered the DAPG's points as MUB projectors (the present paper involves the underpinning of product states by the geometrical points):

$$\alpha = (m, b) \Rightarrow \hat{A}_{m,b} = |m, b\rangle\langle b, m|.$$

This scheme allows relating the underpinning DAPG lines to interrelation among the Hilbert space operators (or states) that form those lines as follows. For $b \neq \ddot{0}$ we have, cf. Eq.(2.3),

$$\langle n | \hat{A}_{m,b} | n' \rangle = \omega^{(n-n')[b/2(n+n'-1)-m]}/d. \quad (5.3)$$

Thus for fixed b , ($\neq \ddot{0}$), and fixed $n \neq n'$ the terms run over the d distinct roots of unity. (Note: $b/2 = b(d+1)/2 \text{ mod}[d]$, as we consider modular numbers.) Thus we have a unique solution, for some m' , given m , to

$$\langle n | \hat{A}_{m,b} | n' \rangle = \langle n | \hat{A}_{m',b'} | n' \rangle, \quad b \neq b', \quad n \neq n'. \quad (5.4)$$

The equality holds whenever, for fixed n, n' $n \neq n'$,

$$\frac{b}{2}(n + n' - 1) - m = \frac{b'}{2}(n + n' - 1) - m'. \quad (5.5)$$

We now assert that that all the d projectors, one for each value of b , with fixed value of $n + n'$ belong to a line. Adjuncted by the projector $|\ddot{m}\rangle\langle\ddot{m}|$ (that belong to the first column, $b = \ddot{0}$), with $2\ddot{m} = n + n'$, the set now forms line. A convenient parametrization for the line obtains upon rearranging Eq.(5.5) and taking $b' = 0$ to get the equation for m as a function of b , i.e. the equation for the line $j = (\ddot{m}, m(0))$, viz. Eq.(5.1).

We now note that projectors $\hat{A}_{m,b}$ that form the line share necessarily all the non diagonal matrix elements $\langle s|\hat{A}_{m,b}|s'\rangle$ with $s + s' = \tilde{m}$ and all the diagonal elements ($=1/d$). This, while matrix elements not abiding with these requirements are distinct. With this we may now evaluate the line operator for this underpinning scheme, [17, 18], noting that for this case, as noted above, the balance formula Eq.(4.3), is $\mathcal{R} = \mathbb{I}$. To illustrate these considerations we evaluate \hat{P}_j , $j = (\tilde{m} = 1, m(0) = 2)$. Via Eq.(5.1) and Eq.(4.4) we have

$$\hat{P}_j = \hat{A}_{(1,\tilde{0})} + \hat{A}_{(2,0)} + \hat{A}_{(1,1)} + \hat{A}_{(0,2)} - \mathbb{I}. \quad (5.6)$$

Via Eq.(5.4),

$$\hat{A}_{(1,\tilde{0})} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{A}_{(2,0)} = \frac{1}{3} \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix}, \quad \hat{A}_{(1,1)} = \frac{1}{3} \begin{pmatrix} 1 & \omega & \omega \\ \omega^2 & 1 & 1 \\ \omega^2 & 1 & 1 \end{pmatrix}, \quad \hat{A}_{(0,2)} = \frac{1}{3} \begin{pmatrix} 1 & 1 & \omega \\ 1 & 1 & \omega \\ \omega^2 & \omega^2 & 1 \end{pmatrix}. \quad (5.7)$$

Eq.(5.6) now gives

$$\hat{P}_j = \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix}. \quad (5.8)$$

The general formula for the matrix elements of the line operator is

$$\langle n|\hat{P}_{j=(\tilde{m},m(0))}|n'\rangle = \delta_{(n+n'),2\tilde{m}}\omega^{-(n-n')m(0)}. \quad (5.9)$$

The proof of this is outlined in Appendix B [17].

This mapping of the Hilbert space projectors onto lines and points of the underpinning geometry was shown in [18] to allow a convenient finite dimensional Radon transform.

VI. GEOMETRIC UNDERPINNING OF TWO-PARTICLES STATES

We now consider DAPG underpinning for *states* of a d^2 dimensional *two* particles, each of d-dimensional Hilbert space. Our coordination scheme is as outlined above $\alpha = (m, b)$; $j = (\tilde{m}, m(0))$, $m(b) = m(0) + b/2(2\tilde{m} - 1)$. However, now each point will refer to a two-particles *state* as is specified below. We have thus,

$$|A\rangle_\alpha; \quad \alpha = 1, 2, \dots, d(d+1), \quad |P\rangle_j; \quad j = 1, 2, \dots, d^2. \quad (6.1)$$

$|A_\alpha\rangle$ are underpinned with the $d(d+1)$ points, S_α while the $|P_j\rangle$ with the d^2 lines, L_j . We define the states, $|A_\alpha\rangle$, underpinned by the geometrical points, by

$$|A_\alpha\rangle \equiv |m, b\rangle_1 |\tilde{m}, \tilde{b}\rangle_2. \quad (6.2)$$

$|\tilde{m}, \tilde{b}\rangle$ is given by Eq.(2.6).

With this we return to states interrelation implied by the geometry: Eqs.(4.1),(4.4) now read

$$|A_\alpha\rangle = \frac{1}{d} \sum_{j \in \alpha} |P_j\rangle \rightarrow |P_j\rangle = \sum_{\alpha \in j} |A_\alpha\rangle - \sum_{\alpha' \in \alpha \cup M_\alpha} |A_{\alpha'}\rangle. \quad (6.3)$$

(note that $|P_j\rangle$ is not normalized.)

We now show that with the choice, $\tilde{m} = d - m$, $\tilde{b} = d - b$, the balance formula, viz the base independence of the balancing term, Eq.(4.3), holds: cf. [12, 20],

$$\begin{aligned} \sum_{\alpha' \in \alpha \cup M_\alpha} |A_{\alpha'}\rangle &\equiv \sum_{m \in b} |A_{m,b}\rangle = \sum_{m \in b} |m, b\rangle_1 |\tilde{m}, \tilde{b}\rangle_2 = \sum_{m, n, n'} |n\rangle_1 |n'\rangle_2 \langle n|m, b\rangle \langle n'|\tilde{m}, \tilde{b}\rangle \\ &= \sum_n |n\rangle_1 |n\rangle_2 = |\mathcal{R}\rangle \quad \text{independent of } b \forall b. \end{aligned} \quad (6.4)$$

This of course includes the first column, $b = \ddot{0}$, with the "point" in the n' row underpinning the state $|n'\rangle_1|n'\rangle_2$.

The relation among the matrix elements of projectors, $\hat{A}_{(m,b)} = |m, b\rangle\langle b, m|$, residing on the line given by Eq.(5.1), [17, 18], with the two particle states, $|A_{(m,b)} = |m, b\rangle_1|\tilde{m}, \tilde{b}\rangle_2$, residing on the equivalent line, Eq.(5.1), are now used to obtain an explicit formula for the line state,

$$\begin{aligned} |P_{j=\ddot{m}, m(0)}\rangle &= \frac{1}{\sqrt{d}} \left(\sum_{m(b) \in j} |m, b\rangle_1 |\tilde{m}, \tilde{b}\rangle_2 - |\mathcal{R}\rangle \right) = \\ &= \frac{1}{\sqrt{d}} \sum_{n, n'} |n\rangle_1 |n'\rangle_2 \left[\langle n| \sum_{m(b) \in j} \hat{A}_{m,b} - \mathbb{I} |n'\rangle \right] = \frac{1}{\sqrt{d}} \sum_{n, n'} |n\rangle_1 |n'\rangle_2 \delta_{n+n', 2\ddot{m}} \omega^{-(n-n')m(0)} \quad \forall b. \end{aligned} \quad (6.5)$$

Where we used Eq(2.6) and Eq.(5.9). The expression for the line state will be put now in a more pliable form,[20],

$$\begin{aligned} |P_{j=\ddot{m}, m(0)}\rangle &= \frac{1}{\sqrt{d}} \sum_{n, n'} |n\rangle_1 |n'\rangle_2 \delta_{n+n', 2\ddot{m}} \omega^{-(n-n')m(0)} = \\ &= \frac{\omega^{2\ddot{m}m(0)}}{\sqrt{d}} \sum_n |n\rangle_1 |2\ddot{m} - n\rangle_2 \omega^{-2nm(0)} = \frac{\omega^{2\ddot{m}m(0)}}{\sqrt{d}} \sum_n |n\rangle_1 \hat{X}_2^{2\ddot{m}} \hat{Z}_2^{2m(0)} \mathcal{I}_2 |n\rangle_2 = \\ &= \frac{\omega^{2\ddot{m}m(0)}}{\sqrt{d}} \sum_m |m, b\rangle_1 \hat{X}_2^{2\ddot{m}} \hat{Z}_2^{2m(0)} \mathcal{I}_2 |\tilde{m}, \tilde{b}\rangle_2. \end{aligned} \quad (6.6)$$

The inversion operator \mathcal{I} is defined via $\mathcal{I}|n\rangle = |-n\rangle = |d-n\rangle$. \hat{X}, \hat{Z} are defined in section II. The orthonormality of $|P_j\rangle$ is proved in appendix C.

The central result of our geometrical underpinning is the following intuitively obvious overlap relation

$$\begin{aligned} \langle A_{\alpha=(m,b)} | P_{j=\ddot{m}, m(0)} \rangle &\equiv \langle m, b |_1 \langle \tilde{m}, \tilde{b} |_2 P_{j=\ddot{m}, m(0)} \rangle = \frac{1}{\sqrt{d}} \delta_{m, (m(0)+b/2[2\ddot{m}-1])}, \quad b \neq \ddot{0}, \\ \langle A_{\alpha=(n,\ddot{0})} | P_{j=\ddot{m}, m(0)} \rangle &\equiv \langle n |_1 \langle n |_2 P_{j=\ddot{m}, m(0)} \rangle = \frac{1}{\sqrt{d}} \delta_{n, \ddot{m}}, \quad b = \ddot{0}, \text{ i.e. computational basis.} \end{aligned} \quad (6.7)$$

Thus the overlap of $|A_{\alpha=(m,b)}\rangle$ with $|P_j\rangle$ vanishes for $\alpha \notin j$ i.e. for $m \neq m(0) + b/2(2\ddot{m} - 1)$: Only if the point (m,b) is on the line j the overlap is non zero. This is a remarkable attribute: Each and every one of the observables $|m, b\rangle_1 |\tilde{m}, \tilde{b}\rangle_2 \langle \tilde{b}, \tilde{m} |_2 \langle b, m |_1$ has a *definite and known* value if measured in the state $|P_{(j, m(0))}\rangle$ yet its constituents single particle observables do *not* commute. Indeed the single particle e.g. $|m, b\rangle_1 \langle b, m |$ has a finite probability to be found anywhere (on every line). The probability of finding our system in the state $|A_\alpha\rangle$ given that the system is in the state $|P_j\rangle$, $\alpha \in j$ is $\frac{1}{d}$. We note, however, that there are $d+1$ points α , exposing that these probabilities are not mutually exclusive. This can be directly checked by noting the non vanishing of the overlap, $|\langle A_\alpha | A_{\alpha'} \rangle| = 1/d$, $\alpha \neq \alpha'$, $\alpha, \alpha' \in j$. The probability when $\alpha \notin j$ is nil. This allows a new approach to the Mean King Problem to which we shall turn after the collective coordinate formulation.

VII. GEOMETRIC VIEW OF COLLECTIVE COORDINATES FORMULATION

The simplification offered by the collective formulation is illustrated by considering the balance term, cf. Eq(4.2), including normalization, Eq.(6.5),

$$\frac{1}{\sqrt{d}} |\mathcal{R}\rangle = \frac{1}{\sqrt{d}} \sum_n |n\rangle_1 |n\rangle_2 = \frac{1}{\sqrt{d}} \sum_n \frac{1}{\sqrt{d}} \sum_{n', n''} |n'\rangle_r |n''\rangle_c \langle n'_r, n''_c | n \rangle_1 |n\rangle_2 = \sum_n |0\rangle_r |n\rangle_c = |0; \ddot{0}\rangle_r |0; 0\rangle_c. \quad (7.1)$$

We used Eq.(3.7) to get $n'_r = 0$, $n''_c = 0$. The RHS reads that within the collective coordinates the state $|\mathcal{R}\rangle$ is a product state: In the r (relative) coordinates it is in computational basis ($b_r = \ddot{0}$) with eigenvalue (of \bar{Z}_r) 1 (i.e. $m=0$). In the c (center of mass) coordinate space it is in $b_c = 0$ with eigenvalue (of \bar{X}_c) 1 too.

We now turn to the expression for $|P_j\rangle$ within the collective coordinates system, using Eq.(6.5),

$$\begin{aligned}
|P_{j=\ddot{m}, m(0)}\rangle &= \frac{1}{\sqrt{d}} \sum_{n, n'} |n\rangle_1 |n'\rangle_2 \delta_{n+n', 2\ddot{m}} \omega^{-(n-n')m(0)} \\
&= \frac{1}{\sqrt{d}} \sum_{n, n'} \sum_{n'_r, n''_c} |n'\rangle_r |n''\rangle_c \langle n'_r, n''_c | n_1, n'_2 \rangle \delta_{n+n', 2\ddot{m}} \omega^{-(n-n')m(0)} \\
&= \frac{1}{\sqrt{d}} \sum_{n_r, n_c} |n\rangle_c |n'\rangle_r \delta_{n_c, \ddot{m}} \omega^{-2n_r m(0)} = |m; \ddot{0}\rangle_c |2m(0); 0\rangle_r.
\end{aligned} \tag{7.2}$$

Identifying the state as a product state in the collective coordinates. (The product state above is notationally simplified by $|\ddot{m}\rangle_c |2m_0\rangle_r$.)

The collective coordinate expression for the particles product state $|A_{\alpha=(m,b)}\rangle$ is,

$$\begin{aligned}
|A_{\alpha=(m,b)}\rangle &= |m, b\rangle_1 |\tilde{m}, \tilde{b}\rangle_2 = \frac{1}{d} \sum_{n, n'} |n\rangle_1 |n'\rangle_2 \omega^{(n-n')[\frac{b}{2}(n+n'-1)-m]}, \\
&= \sum_{k, k'} |k\rangle_r |k'\rangle_c \omega^{2k[\frac{b}{2}(2k'-1)-m]}.
\end{aligned} \tag{7.3}$$

The probability amplitude of finding the particles in the state $|A_{\alpha=(m,b)}\rangle$ given a system in the state $|\ddot{m}; \ddot{0}\rangle_c |2m(0); 0\rangle_r \equiv |\ddot{m}\rangle_c |m_0\rangle_r$ is

$$\begin{aligned}
\frac{1}{d} \sum_{k, k'} \langle k | 2m(0); 0 \rangle \langle k' | \ddot{m} \rangle \omega^{2k[\frac{b}{2}(2k'-1)-m]} &= \frac{1}{d^{\frac{3}{2}}} \sum_k \sum_n \langle k | n \rangle \omega^{-2nm(0)} \omega^{2k[\frac{b}{2}(2\ddot{m}-1)-m]} = \frac{\delta_{m, (m(0)+\frac{b}{2}(2\ddot{m}-1))}}{\sqrt{d}}; \quad b \neq \ddot{0} \\
\cdot \langle n | 1 \rangle \langle n | 2m(0); 0 \rangle_r | \ddot{m} \rangle_c &= \frac{\delta_{n, \ddot{m}}}{\sqrt{d}}, \quad b = \ddot{0}.
\end{aligned} \tag{7.4}$$

Thus the *probability* is $\frac{1}{d}$ if the state is on the line (nil if it is not), confirming Eq.(6.7) and the efficiency of the collective coordinate formulation.

VIII. LEAKY PARTICLES

The maximally entangled state, Eq.(7.2),

$$|P_{j=(\ddot{m}, m_0)}\rangle \equiv |\ddot{m}\rangle_c |2m_0\rangle_r,$$

was viewed as a "line" state. I.e. the product states underpinned by the geometrical point, $\alpha = (m, b)$,

$$|A_\alpha\rangle = |m, b\rangle_1 |\tilde{m}, \tilde{b}\rangle_2,$$

whose coordinates, (m,b), abides by the line equation, Eq. (5.1),

$$m(b) = m_0 + \frac{b}{2}(2\ddot{m} - 1), \quad b \neq \ddot{0}, \quad m(\ddot{0}) = \ddot{m}$$

form form line in the sense that (cf. Eq.(VI)),

$$\langle A_\alpha | P_j \rangle \equiv \langle \tilde{b}, \tilde{m} | 2 \langle b, m | 1 \rangle \ddot{m} \rangle_c | 2m_0 \rangle_r = \frac{1}{d}, \quad \alpha \in j, \text{ nonumber} \tag{8.1}$$

$$= 0 \quad \alpha \notin j. \tag{8.2}$$

Thus a pair of particles (the particle and its mate, the tilde particle) whose coordinates are $\alpha = m, b$ do wholly belong to the d lines that share the coordinated point. However each of the constituent particles (either 1 or 2) is *equally* likely to be in anyone the d^2 of the lines,

$$\langle b, m|_1 \ddot{m} \rangle_c |2m_0\rangle = \frac{1}{\sqrt{d}} |(\ddot{m} - 2\tilde{\Delta}), \tilde{b}\rangle_2 \omega^{2\ddot{m}\Delta}, \quad \Delta = m - m_0 + \frac{b}{2}(2\ddot{m} - 1), \quad (8.3)$$

($\tilde{\Delta} = d - \Delta$.) Thence,

$$|\langle b, m|_1 \ddot{m} \rangle_c |2m_0\rangle|^2 = \frac{1}{d}, \quad \forall(\ddot{m}, m_0).$$

It is this attribute that allows the tracking of the King measurement alignment.

IX. TRACKING THE MEAN KING

The Mean King Problem (MKP), initiated by [13], was analyzed in several publications - see the comprehensive list in [12]. Briefly summarized it runs as follows. Alice may prepare a state to her liking. The King measures it in an MUB basis (i.e. for some value of b : a particular alignment of his apparatus). He does not inform Alice of his observational result nor the basis he used. Alice performs a control measurement of her choice. *After* her control measurement the King informs her the basis, b , he used for his measurement. Thence she must *deduce* the actual state (m, b) that he observed. In our case of *tracking* the King - He does not inform Alice of the basis he used - her control measurement is designed to track the basis used. (Note that in all the analyses time evolution is ignored - presumed to be independently accountable.)

The state that Alice prepares is one of the line vectors,

$$|P_{j=(\ddot{m}, m(0))}\rangle = |\ddot{m}\rangle_c |2m_0\rangle_r$$

. Thus she knows both \ddot{m} and $m(0)$. The King's measurement is along a line of some fixed b i.e. he measures

$$\sum_m |m, b\rangle \omega^m \langle b, m|$$

and observed, say, ω^m . The King's measurement projects the state $|P_j\rangle$ to (neglecting normalization):

$$|m, b\rangle_1 \langle b, m|_1 |\ddot{m}\rangle_c |2m_0\rangle_r.$$

Now Alice measures the *non degenerate* operator,

$$\sum_{m', m''} |\ddot{m}'\rangle_c |2m_0''\rangle_r \gamma_{m', m''} \langle 2m_0'' | \langle \ddot{m}'|,$$

and obtains, say, $\gamma_{m', m''}$. Thence the quantity,

$$\langle 2m_0'' |_r \langle \ddot{m}' |_c m, b \rangle_1 \langle b, m|_1 |\ddot{m}\rangle_c |2m_0\rangle_r \neq 0. \quad (9.1)$$

The LHS of this equation is obtained by using Eq.(8.3) and noting that

$$\langle 2m_0'' |_r \langle \ddot{m}' |_c m, b \rangle_1$$

may be obtained from it by replacing \ddot{m}, m_0 by \ddot{m}', m_0' and taking the complex conjugate expression. The result is,

$$\langle 2m_0'' |_r \langle \ddot{m}' |_c m, b \rangle_1 \langle b, m|_1 |\ddot{m}\rangle_c |2m_0\rangle_r = \frac{1}{d} \delta[(m_0' - m_0), b(\ddot{m} - \ddot{m}')]. \quad (9.2)$$

i.e.

$$b = \frac{(m_0' - m_0)}{(\ddot{m} - \ddot{m}')}; \quad \ddot{m} = \ddot{m}' \rightarrow b = \ddot{0}. \quad (9.3)$$

Knowing the initial state, i.e. \ddot{m} and m_0 and measuring the final state, viz. \ddot{m}' and m_0' Alice tracks b - the apparatus alignment used by the King. (The alignment $b = \ddot{0}$ - the King's CB - gives $\delta_{\ddot{m}, \ddot{m}'}$.)

X. SUMMARY AND CONCLUDING REMARKS

We outlined finite plane geometrical underpinning to *states* of two d-dimensional particles Hilbert spaces. $d=\text{prime}$ ($\neq 2$). The geometrical points α , were coordinated with mutual unbiased bases (MUB) states labels: $\alpha = (m, b)$. m denotes the vector in the base b , it locates the vertical coordinate within the column labelled by b . b gives the position of the column. Points of the geometry $\alpha = (m, b)$ underpin product states labelled with (m, b) : $|A_\alpha\rangle = |m, b\rangle_1 |\tilde{m}, \tilde{b}\rangle_2$, 1 and 2 refers to the particles with $\tilde{m} = d - m$, $\tilde{b} = d - b$. A geometrical line, j , runs through $d+1$ points (one point at each MUB basis) - it underpins the state vector $|P_{j=(\tilde{m}, m_0)}\rangle$. This vector is completely parametrized by two points $(m(\vec{0}), m(0)) \equiv (\tilde{m}, m_0)$ - the point on the computational basis column ($b = \vec{0}$) and the point on the $b=0$ column i.e. the eigenfunctions of the displacement operator. Viewing a line as a point's evolution with increasing b , [21], the line label specified with \tilde{m} and $m(0)$ may be viewed as specification in terms of initial position and momentum [22]. The states $|P_j\rangle$, $j = 1, 2, \dots, d^2$ are maximally entangled and form an orthonormal basis that spans the space. These states have a remarkable attribute: the probability of finding it in the state $|A_\alpha\rangle$, $\alpha \in j$ is $1/d$, it vanishes otherwise. The number of points α , on a line is $d+1$ - reflecting the *non-exclusiveness* of these probabilities which, in turn, allows the tracking of the King measurement as is accounted above.

We gave an alternative, perhaps more economic, parametrization of the d^2 *maximally entangled* states that span the space - parametrization based on a collective, viz center of mass and relative, coordinates. Here the state vector underpinned with a geometrical line is given by a *product* state of the collective coordinates, $|P_{j=(\tilde{m}, m_0)}\rangle = |\tilde{m}\rangle_c |2m_0\rangle_r$ (c and r are center of mass and relative coordinates respectively). These states were shown to provide simplified notation for the calculation as well as a novel view of maximally entangled states.

It was shown that adding up product states in geometrically reasoned manner yields maximally entangled states. These states are shown to be product states of two particles collective (center of mass and relative) coordinates. The states are such as to allow unambiguous tracking of alignment of measurement of their constituent single particle.

Appendix A: Geometrically based Hilbert space operators' interrelation

Our task is to define consistently addition (and subtractions) of "line" and "point" Hilbert space operators (or states) which are underpinned by geometrical points and lines assuring that they abide by their geometrical underpinning interrelation. The logical interrelation symbols (S and L represents the geometrical point and line respectively),

$$\bigcup_{\alpha \in j} S_\alpha = L_j; \quad \bigcap_{j \in \alpha} L_j = S_\alpha$$

are to be realized by addition (and subtraction) of Hilbert space entities, operators or states, supplemented with numerical values. Our starting point is :

$$S_\alpha = \bigcap_{j \in \alpha} L_j \Rightarrow A_\alpha = \frac{1}{d} \sum_{j \in \alpha} P_j, \quad (10.1)$$

where we underpinned the Hilbert space operator (or state) A_α with the point S_α . We now consider a particular realization of the geometry, i.e. a set up where the points and lines abide by the geometry are realized by marking the points on each line subject to DAPG requirements such as, e.g., two distinct lines have a single point in common. The geometry is then realized via Eq.(10.1), coordinated as specified by MUB labelings. It, then, follows via DAPG(a,c,d,e) - cf. Eq.(4.2), that

$$\sum_{\alpha' \in \alpha \cup M_\alpha} A_{\alpha'} = \frac{1}{d} \sum_{\alpha' \in \alpha \cup M_\alpha} \left(\sum_{j \in \alpha'} P_j \right) = \frac{1}{d} \sum_{j=1}^{d^2} P_j. \quad (10.2)$$

The RHS is clearly a universal quantity (i.e. independent of α and j) which implies that the LHS,

$$\sum_{\alpha \in \alpha' \cup M_{\alpha'}} A_\alpha = \mathcal{R}, \text{ independent of } \alpha',$$

i.e. universal too.

Since a line is made of points we consider (try)

$$P_j = \sum_{\alpha \in j} A_\alpha - \mathcal{R}, \quad (10.3)$$

where \mathcal{R} is a universal quantity that may be required to balance the equation. Returning to Eq.(10.1) with Eq. (10.3), *the geometry* implies via DAPG(c,d),

$$A_\alpha = A_\alpha + d \sum_{\alpha \in \alpha' \cup M_{\alpha'}} A_\alpha - d\mathcal{R}; \Rightarrow \sum_{\alpha \in \alpha' \cup M_{\alpha'}} A_\alpha = \mathcal{R}. \quad (10.4)$$

We illustrate the consistency of this by showing the validity of the geometrically derived reation, Eq.(4.3):

$$\begin{aligned} \frac{1}{d} \sum_j \frac{d^2}{d} P_j &= \frac{1}{d} \sum_j \left(\sum_{\alpha \in j} A_\alpha - \mathcal{R} \right) = -d\mathcal{R} + \sum_{\alpha} A_\alpha = \\ &= -d\mathcal{R} + (d+1) \sum_{\alpha' \in \alpha \cup M_\alpha} A_{\alpha'} = \mathcal{R}. \quad QED. \end{aligned}$$

Where we used the universality of \mathcal{R} and DAPG(c,d). Thus

$$P_j = \sum_{\alpha \in j} A_\alpha - \mathcal{R}.$$

With,

$$\begin{aligned} A_\alpha &= |m, b\rangle \langle b, m| \Rightarrow \hat{\mathcal{R}} = \hat{\mathbb{I}}; \\ A_\alpha &= |m, b\rangle_1 \langle \tilde{m}, \tilde{b}|_2 \Rightarrow |\mathcal{R}\rangle = \sum_m |m, b\rangle_1 \langle \tilde{m}, \tilde{b}|_2. \end{aligned} \quad (10.5)$$

Distintict consistent constructions are possible,[11], for example with a starting point,

$$\hat{P}_j = \frac{1}{d+1} \sum_{\alpha \in j} \hat{A}_\alpha, \quad (10.6)$$

one is lead to

$$\hat{A}_\alpha = \frac{d+1}{d} \sum_{j \in \alpha} \hat{P}_j - \hat{\mathcal{R}}.$$

Appendix B: The Line operator \hat{P}

With the geometrical point, $\alpha = (m, b)$, underpinning the MUB *projector* $\hat{A}_{\alpha=m,b} = |m, b\rangle \langle b, m|$ the geometrical line $j = (\ddot{m}, m_0)$ underpins the operator \hat{P}_j , Eq. (5.3,4.4, 5.6),

$$\hat{P}_j = \sum_{\alpha \in j} \hat{A}_\alpha - \mathbb{I}. \quad \langle n | \hat{A}_{\alpha=(m,b)} | n' \rangle = \frac{\omega^{(n-n')[\frac{b}{2}(n+n'-1)-m]}}{d}.$$

A line \hat{P}_j is given by the contribution of point projectors \hat{A}_α with common matrix elements, i.e. \hat{A}_α and $\hat{A}_{\alpha'}$ belong to the same line whenever $\langle n | \hat{A}_\alpha | n' \rangle = \langle n | \hat{A}_{\alpha'} | n' \rangle$ (for $b \neq 0$). This reads for $n+n'$ =common constant which

we chose to be $2\ddot{m}$ - which a point on the line on the $b = \ddot{0}$ column. The next point on the line is the value of m on the $b=0$ column, $m(0) = m_0$. The line is now defined by the two points $j = (\ddot{m}, m_0)$. All other points are now given by

$$-m(0) = \frac{b}{2}(2\ddot{m} - 1) - m(b) \rightarrow m(b) = \frac{b}{2}(2\ddot{m} - 1) + m_0.$$

Since all the other matrix elements of the projectors forming the line are distinct and are, each, a d -root of unity their sum add up to zero. Thus the final formula for \hat{P}_j is

$$\langle n | \hat{P}_j | n' \rangle = \delta_{(n+n'), 2\ddot{m}} \omega^{-(n-n')m_0}. \quad QED$$

Appendix C: Orthogonality of $|P_j\rangle$

Noting that $\langle \mathcal{R} | \mathcal{R} \rangle = d$ and $\langle P_j | \mathcal{R} \rangle = d + 1$, We get, for $j=j'$:

$$\langle P_j | P_j \rangle = \frac{1}{d} \left\{ \sum_{\alpha \in j} \langle A_\alpha | - \langle \mathcal{R} | \right\} \left\{ \sum_{\alpha' \in j} | A_{\alpha'} \rangle - | \mathcal{R} \rangle \right\} = \frac{1}{d} \{ 1 + d + \sum_{\alpha \neq \alpha'}^{d+1} [\langle A_\alpha | A_{\alpha'} \rangle] - 2(d+1) + d \} = 1 \quad (10.7)$$

Where we used that $\sum_{\alpha \neq \alpha'}^{d+1} [\langle A_\alpha | A_{\alpha'} \rangle] = (d+1) \frac{d}{d} = d + 1$.

For $j \neq j'$ the geometry dictates, DAPG(b), that distinct lines share *one* point. Thus the first term above is 1 rather than $1+d$ hence

$$\langle P_j | P_{j'} \rangle = 0, \quad j \neq j'$$

i.e.

$$\langle P_j | P_{j'} \rangle = \delta_{j,j'}. \quad QED \quad (10.8)$$

Note: Using the collective coordinates, $|P_{j=(\ddot{m}, m(0))}\rangle = |\ddot{m}\rangle_c |m_0\rangle_r$, the proof is immediate.

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